8.2 Matrices

8.2.1 Definition

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. The numbers are called the elements of the matrix or entries in the matrix. A matrix is represented by capital letters *A, B, C etc*. and its elements by small letters *a,b,c,x,y etc*. The following are some examples of matrices:

$$A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix},\ B=\begin{bmatrix}2+i & -3 & 2\\ 1 & -3+i & -5\end{bmatrix},\ C=[1,\ 4,\ 9],\ \ D=\begin{bmatrix}a\\ g\\ h\end{bmatrix},\ \ E=[l]$$

8.2.2 Order of a Matrix

A matrix having m rows and n columns is called a matrix of order $m\times n$ or simply $m\times n$ matrix (read as 'an m by n matrix). A matrix A of order $m\times n$ is usually written in the following manner

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13} & ...a_{1j} & ...a_{1n}\\ a_{21} & a_{22} & a_{23} & ...a_{2j} & ...a_{2n}\\ ..... & ..... & ..... & ..... & .....\\ a_{i1} & a_{i2} & a_{i3} & ...a_{ij} & ...a_{in}\\ ..... & ..... & ..... & ..... & .....\\ a_{m1} & a_{m2} & a_{m3} & ...a_{mj} & ...a_{mn}\end{bmatrix} \text{ or } A=[a_{ij}]_{m\times n}, \text{ where } \begin{matrix} i=1,\ 2,.....m\\ j=1,\ 2,.....n\end{matrix}$$

Here a_{ij} denotes the element of ith row and jth column. *Example* : order of matrix $\begin{bmatrix}3 & -1 & 5\\ 6 & 2 & -7\end{bmatrix}$ is 2×3

Note : ❑ A matrix of order $m\times n$ contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

8.2.3 Equality of Matrices

Two matrix A and B are said to be equal matrix if they are of same order and their corresponding elements are equal *Example*: If $A=\begin{bmatrix}1 & 6 & 3\\ 5 & 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}$ are equal matrices.

Then $a_1=1, a_2=6, a_3=3, b_1=5, b_2=2, b_3=1$

8.2.4 Types of Matrices

(1) **Row matrix** : A matrix is said to be a row matrix or row vector if it has only one row and any number of columns. *Example* : [5 0 3] is a row matrix of order 1× 3 and [2] is a row matrix of order 1×1.

(2) **Column matrix** : A matrix is said to be a column matrix or column vector if it has only one column and any number of rows. *Example* : $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3×1 and [2] is a column matrix of order 1×1. Observe that [2] is both a row matrix as well as a column matrix.

(3) **Singleton matrix** : If in a matrix there is only one element then it is called singleton matrix.

Thus, $A = [aij]_{m \times n}$ is a singleton matrix if $m = n = 1$ *Example* : [2], [3], [a], [–3] are singleton matrices.

(4) **Null or zero matrix** : If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by *O*. Thus $A = [a_{ij}]_{m \times n}$ is a zero matrix if $a_{ij} = 0$ for all *i* and *j*.

Example : $[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [0\ 0]$ are all zero matrices, but of different orders.

(5) **Square matrix** : If number of rows and number of columns in a matrix are equal, then it is called a square matrix. Thus $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$. *Example* : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3

(i) If $m \neq n$ then matrix is called a rectangular matrix.

(ii) The elements of a square matrix A for which $i = j$, *i.e.* $a_{11}, a_{22}, a_{33}, \ldots. a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix *A*.

(iii) **Trace of a matrix :** The sum of diagonal elements of a square matrix. *A* is called the trace of matrix A , which is denoted by tr A. $tr\ A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots a_{nn}$

Properties of trace of a matrix : Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar

(i) $tr(\lambda A) = \lambda\, tr(A)$ (ii) $tr(A - B) = tr(A) - tr(B)$ (iii) $tr(AB) = tr(BA)$

(iv) $tr(A) = tr(A')$ or $tr\ (A^T)$ (v) $tr(I_n) = n$ (vi) tr (0)= 0

(vii) $tr(AB) \neq tr\,A\,.\,tr\,B$

(6) **Diagonal matrix** : If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3×3, which can be denoted by diag [2, 3, 4]

Note : ❑ No element of principal diagonal in a diagonal matrix is zero.

❑ Number of zeros in a diagonal matrix is given by $n^2 - n$ where n is the order of the matrix.

❑ A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots\ldots, d_n$ as diagonal elements is denoted by $\text{diag}\ [d_1, d_2, \ldots, d_n]$.

(7) **Identity matrix** : A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. Thus, the square matrix $A = [a_{ij}]$ is an identity matrix, if $a_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$

We denote the identity matrix of order n by I_n.

Example : [1], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.

(8) **Scalar matrix** : A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix. Thus, if $A = [a_{ij}]$ is a square matrix and $a_{ij} = \begin{cases} \alpha, \text{if } i = j \\ 0, \text{ if } i \neq j \end{cases}$, then A is a scalar matrix.

Example : [2], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ are scalar matrices of order 1, 2 and 3 respectively.

Note : ❑ Unit matrix and null square matrices are also scalar matrices.

(9) **Triangular Matrix** : A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types

(i) **Upper Triangular matrix :** A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij} = 0$ when $i > j$.

Example : $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix of order 3×3.

(ii) **Lower Triangular matrix :** A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$.

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3×3.

Note : ❑ Minimum number of zeros in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

❑ Diagonal matrix is both upper and lower triangular.

❑ A triangular matrix $a=[a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ij}=0$ for $1\le i\le n$

Example: 1 A square matrix $A=[a_{ij}]$ in which $a_{ij}=0$ for $i\ne j$ and $a_{ij}=k$ (constant) for $i=j$ is called a

(a) Unit matrix (b) Scalar matrix (c) Null matrix (d) Diagonal matrix

Solution: (b) When $a_{ij}=0$ for $i\ne j$ and a_{ij} is constant for $i=j$ then the matrix $[a_{ij}]_{n\times n}$ is called a scalar matrix

Example: 2 If *A, B* are square matrix of order 3, A is non singular and $AB=0$, then *B* is a

(a) Null matrix (b) Singular matrix (c) Unit matrix (d) Non singular matrix

Solution: (a) *AB* = 0 when *B* is null matrix.

Example: 3 The matrix $\begin{bmatrix}2 & 5 & -7\\ 0 & 3 & 11\\ 0 & 0 & 9\end{bmatrix}$ is known as

(a) Symmetric matrix (b) Diagonal matrix (c) Upper triangular matrix (d)Skew symmetric matrix

Solution: (c) We know that if all the elements below the diagonal in a matrix are zero, then it is an upper triangular matrix.

Example: 4 In an upper triangular matrix *n*×*n*, minimum number of zeros is **[Rajasthan PET 1999]**

(a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+1)}{2}$ (c) $\frac{2n(n-1)}{2}$ (d) None of these

Solution: (a) As we know a square matrix $A=[a_{ij}]$ is called an upper triangular matrix if $a_{ij}=0$ for all *i*>*j*

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14} & .... & a_{1(n-2)} & a_{1(n-1)} & a_{1n}\\ 0 & a_{22} & a_{23} & a_{24} & .... & a_{2(n-2)} & a_{2(n-1)} & a_{2n}\\ 0 & 0 & a_{33} & a_{34} & .... & a_{3(n-2)} & a_{3(n-1)} & a_{3n}\\ 0 & 0 & 0 & a_{44} & .... & a_{4(n-2)} & a_{4(n-1)} & a_{4n}\\ - & - & - & - & - & - & - & -\\ - & - & - & - & - & - & - & -\\ 0 & 0 & 0 & 0 & .... & 0 & a_{(n-1)(n-1)} & a_{(n-1)n}\\ 0 & 0 & 0 & 0 & .... & 0 & 0 & a_{nn}\end{bmatrix}.$$ Number of zeros

$=(n-1)+(n-2)+.....+2+1=\frac{(n-1)n}{2}$

Example: 5 If $A=[a_{ij}]$ is a scalar matrix then trace of *A* is

(a) $\sum_i\sum_j a_{ij}$ (b) $\sum_i a_{ij}$ (c) $\sum_j a_{ij}$ (d) $\sum_i a_{ii}$

Solution: (d) The trace of $A=\sum_{i=1}^{n} a_{ii}$ = Sum of diagonal elements.

8.2.5 Addition and Subtraction of Matrices

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order then their sum *A*+*B* is a matrix whose each element is the sum of corresponding elements. *i.e.* $A+B=[a_{ij}+b_{ij}]_{m\times n}$

Example : If $A=\begin{bmatrix}5 & 2\\ 1 & 3\\ 4 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 5\\ 2 & 2\\ 3 & 3\end{bmatrix}$, then $A+B=\begin{bmatrix}5+1 & 2+5\\ 1+2 & 3+2\\ 4+3 & 1+3\end{bmatrix}=\begin{bmatrix}6 & 7\\ 3 & 5\\ 7 & 4\end{bmatrix}$

Similarly, their subtraction $A-B$ is defined as $A-B=[a_{ij}-b_{ij}]_{m\times n}$

i.e. in above example $A-B=\begin{bmatrix}5-1 & 2-5\\1-2 & 3-2\\4-3 & 1-3\end{bmatrix}=\begin{bmatrix}4 & -3\\-1 & 1\\1 & -2\end{bmatrix}$

Note : ❑ Matrix addition and subtraction can be possible only when matrices are of the same order.

Properties of matrix addition : If *A, B* and *C* are matrices of same order, then

(i) $A+B=B+A$ (Commutative law)

(ii) $(A+B)+C=A+(B+C)$ (Associative law)

(iii) $A+O=O+A=A$, where O is zero matrix which is additive identity of the matrix.

(iv) $A+(-A)=0=(-A)+A$, where $(-A)$ is obtained by changing the sign of every element of *A*, which is additive inverse of the matrix.

(v) $\left.\begin{array}{l}A+B=A+C\\B+A=C+A\end{array}\right\}\Rightarrow B=C$ (Cancellation law)

8.2.6 Scalar Multiplication of Matrices

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be a number, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.

Thus, if $A=[a_{ij}]_{m\times n}$, then $kA=Ak=[ka_{ij}]_{m\times n}$. *Example* : If $A=\begin{bmatrix}2 & 4\\3 & 1\\4 & 6\end{bmatrix}$, then $5A=\begin{bmatrix}10 & 20\\15 & 5\\20 & 30\end{bmatrix}$

Properties of scalar multiplication:

If A, B are matrices of the same order and λ, μ are any two scalars then

(i) $\lambda(A+B)=\lambda A+\lambda B$

(ii) $(\lambda+\mu)A=\lambda A+\mu A$

(iii) $\lambda(\mu A)=(\lambda\mu A)=\mu(\lambda A)$

(iv) $(-\lambda A)=-(\lambda A)=\lambda(-A)$

Note : ❑ All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

8.2.7 Multiplication of Matrices

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in *B* (post multiplier).Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are two matrices of order $m\times n$ and $n\times p$ respectively, then their product *AB* is of order $m\times p$ and is defined as $(AB)_{ij}=\sum_{r=1}^{n}a_{ir}b_{rj}$

$=[a_{i1}a_{i2}...a_{in}]\begin{bmatrix}b_{1j}\\b_{2j}\\\vdots\\b_{nj}\end{bmatrix}$ = (i^{th} row of A)(j^{th} column of B) .....(i), where i=1, 2, ..., m and j=1, 2, ...p

Now we define the product of a row matrix and a column matrix.

Let $A=[a_1 a_2 ....a_n]$ be a row matrix and $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column matrix.

Then $AB=[a_1b_1+a_2b_2+....+a_nb_n]$ ...(ii). Thus, from (i),

$(AB)_{ij}=$ Sum of the product of elements of i^{th} row of A with the corresponding elements of j^{th} column of B.

Properties of matrix multiplication

If A,B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB)C=A(BC)$ (Associative Law)

(iii) $IA=A=AI$, where I is identity matrix for matrix multiplication

(iv) $A(B+C)=AB+AC$ (Distributive law)

(v) If $AB=AC \not\Rightarrow B=C$ (Cancellation law is not applicable)

(vi) If $AB=0$ It does not mean that $A=0$ or $B=0$, again product of two non zero matrix may be a zero matrix.

Note :
- ❑ If A and B are two matrices such that AB exists, then BA may or may not exist.
- ❑ The multiplication of two triangular matrices is a triangular matrix.
- ❑ The multiplication of two diagonal matrices is also a diagonal matrix and

$$\text{diag }(a_1,a_2,....a_n)\times \text{diag }(b_1,b_2,....b_n)=\text{diag }(a_1b_1,a_2b_2,....a_nb_n)$$

- ❑ The multiplication of two scalar matrices is also a scalar matrix.
- ❑ If A and B are two matrices of the same order, then

(i) $(A+B)^2=A^2+B^2+AB+BA$ (ii) $(A-B^2)=A^2+B^2-AB-BA$

(iii) $(A-B)(A+B)=A^2-B^2+AB-BA$ (iv) $(A+B)(A-B)=A^2-B^2-AB+BA$

(v) $A(-B)=(-A)B=-(AB)$

8.2.8 Positive Integral Powers of A Matrix

The positive integral powers of a matrix A are defined only when A is a square matrix. Also then $A^2=A.A$, $A^3=A.A.A=A^2A$. Also for any positive integers m ,n.

(i) $A^mA^n=A^{m+n}$ (ii) $(A^m)^n=A^{mn}=(A^n)^m$

(iii) $I^n=I, I^m=I$ (iv) $A^0=I_n$ where A is a square matrix of order n.

8.2.9 Matrix Polynomial

Let $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_{n-1}x+a_n$ be a polynomial and let A be a square matrix of order n. Then $f(A)=a_0A^n+a_1A^{n-1}+a_2A^{n-2}+...+a_{n-1}A+a_nI_n$ is called a matrix polynomial.

Example : If $f(x)=x^2-3x+2$ is a polynomial and A is a square matrix, then $A^2-3A+2I$ is a matrix polynomial.

Example: 6 If $A=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $A^2=$ **[Rajasthan PET 2001]**

(a) $\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$ (b) $\begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$ (c) $\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$ (d) $\begin{bmatrix} -\cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & -\cos 2\alpha \end{bmatrix}$

Solution: (c) Since $A^2 = A.A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$

Example: 7 If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ then **[AIEEE 2003]**

(a) $\alpha = a^2 + b^2, \beta = ab$ (b) $\alpha = a^2 + b^2, \beta = 2ab$ (c) $\alpha = a^2 + b^2, \beta = a^2 - b^2$ (d) $\alpha = 2ab, \beta = a^2 + b^2$

Solution: (b) $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$. On comparing, we get, $\alpha = a^2 + b^2, \beta = 2ab$

Example: 8 If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, n \in N$, then A^{4n} equals **[AMU 1992]**

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ (c) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution: (a) $A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^4 = A^2.A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$; $(A^4)^n = I^n = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example: 9 If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$ then value of a and b are **[Kurukshetra CEE 2002]**

(a) $a = 4, b = 1$ (b) $a = 1, b = 4$ (c) $a = 0, b = 4$ (d) $a = 2, b = 4$

Solution: (b) We have $(A+B)^2 = A^2 + B^2 + A.B + BA$

$\because AB + BA = 0$ $\therefore \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} + \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} = 0$

$\begin{bmatrix} 2a+2-b & -a+1 \\ 2a-2 & 4-b \end{bmatrix} = 0$. On comparing, we get, $-a+1=0 \Rightarrow a=1$ and $4-b=0 \Rightarrow b=4$

Example: 10 The order of $[x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is **[EAMCET 1994]**

(a) 3×1 (b) 1×1 (c) 1×3 (d) 3×3

Solution: (b) Order will be $(1\times 3)(3\times 3)(3\times 1) = (1\times 1)$

Example: 11 Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $F(\alpha).F(\alpha')$ is equal to

(a) $F(\alpha\alpha')$ (b) $F(\alpha/\alpha')$ (c) $F(\alpha+\alpha')$ (d) $F(\alpha-\alpha')$

Solution: (c) We have $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F(\alpha') = \begin{bmatrix} \cos\alpha' & -\sin\alpha' & 0 \\ \sin\alpha' & \cos\alpha' & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$F(\alpha).F(\alpha') = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha' & -\sin\alpha' & 0 \\ \sin\alpha' & \cos\alpha' & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\alpha') & -\sin(\alpha+\alpha') & 0 \\ \sin(\alpha+\alpha') & \cos(\alpha+\alpha') & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(\alpha+\alpha')$$

Example: 12 For the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, which of the following is correct

(a) $A^3 + 3A^2 - I = 0$ (b) $A^3 - 3A^2 - I = 0$ (c) $A^3 + 2A^2 - I = 0$ (d) $A^3 - A^2 + I = 0$

Solution: (b) $A^2 = A.A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 2 \\ 3 & 4 & 1 \end{bmatrix}$, $A^3 = A.^2A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 2 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 3 \\ 15 & 19 & 6 \\ 9 & 12 & 4 \end{bmatrix}$

$$A^3 - 3.A^2 = \begin{bmatrix} 7 & 9 & 3 \\ 15 & 19 & 6 \\ 9 & 12 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 9 & 3 \\ 15 & 18 & 6 \\ 9 & 12 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \Rightarrow A^3 - 3A^2 - I = 0$$

Example: 13 If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then the value of α for which $A^2 = B$ is **[IIT Screening 2003]**

(a) 1 (b) –1 (c) 4 (d) No real values

Solution: (d) $A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix}$ $\because$ $A^2 = B$ (given)

Then $\begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \Rightarrow \alpha^2 = 1$ and $\alpha + 1 = 5$. Clearly no real value of α

8.2.10 Transpose of a Matrix

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of Matrix A and is denoted by A^T or A'.

From the definition it is obvious that if order of A is $m \times n$, then order of A^T is $n \times m$

Example : Transpose of matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2\times 3}$ is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3\times 2}$

Properties of transpose : Let A and B be two matrices then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$, A and B being of the same order

(iii) $(kA)^T = kA^T$, k be any scalar (real or complex)

(iv) $(AB)^T = B^T A^T$, A and B being conformable for the product AB

(v) $(A_1 A_2 A_3 ..... A_{n-1} A_n)^T = A_n{}^T A_{n-1}{}^T .... A_3{}^T A_2{}^T A_1{}^T$

(vi) $I^T = I$

8.2.11 Determinant of a Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by $|A|$ or Det (A) is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Properties of determinant of a matrix

(i) $|A|$ exists $\Leftrightarrow A$ is square matrix (ii) $|AB| = |A||B|$

(iii) $|A^T| = |A|$ (iv) $|kA| = k^n |A|$, if A is a square matrix of order n

(v) If A and B are square matrices of same order then $|AB| = |BA|$

(vi) If A is a skew symmetric matrix of odd order then $|A| = 0$

(vii) If $A = \text{diag}\,(a_1, a_2, ...a_n)$ then $|A| = a_1 a_2 ... a_n$ (viii) $|A|^n = |A^n|$, $n \in N$.

Example: 14 If A and B are square matrices of same order then **[Pb. CET 1992; Roorkee 1995; MP PET 1990; Rajasthan PET 1992, 94]**

(a) $(AB)' = A'B'$ (b) $(AB)' = B'A'$

(c) $AB = 0$, if $|A| = 0$ or $|B| = 0$ (d) $AB = 0$, if $|A| = I$ or $B = I$

Solution: (b) $A = [a_{ij}]_{n\times n}, B = [b_{jk}]_{n\times n}$, $AB = [a_{ij}]_{n\times n}[b_{jk}]_{n\times n} = [c_{ik}]_{n\times n}$, where $c_{ik} = a_{ij}b_{jk}$

$(AB)' = [c_{ik}]'_{n\times n} = [c_{ki}]_{n\times n} = [b_{kj}]_{n\times n}[a_{ji}]_{n\times n} = B'A'$

Alternatively, Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}, B = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}_{2\times 2}$; $AB = \begin{bmatrix} 1 & 11 \\ 3 & 25 \end{bmatrix}$

$(AB)' = \begin{bmatrix} 1 & 3 \\ 11 & 25 \end{bmatrix}$ .....(i) and $B'A' = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 11 & 25 \end{bmatrix}$ .....(ii)

From (i) and (ii), $(AB) = B'A'$

Example: 15 If A,B are 3×2 order matrices and C is a 2×3 order matrix, then which of the following matrices not defined **[Rajasthan PET 1998]**

(a) $A^t + B$ (b) $B + C^t$ (c) $A^t + C$ (d) $A^t + B^t$

Solution: (a) Order of A is 3 × 2 and order of B is 3 × 2 and order of A^t is 2 × 3 then

$= A^t + B = [A^t]_{2\times 3} + [B]_{3\times 2}$ is not possible because order are not same.

8.2.12 Special Types of Matrices

(1) Symmetric and skew-symmetric matrix

(i) **Symmetric matrix** : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all *i, j* or $A^T = A$

Example : $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Note : ❑ Every unit matrix and square zero matrix are symmetric matrices.

❑ Maximum number of different elements in a symmetric matrix is $\frac{n(n+1)}{2}$

(ii) **Skew-symmetric matrix** : A square matrix $A = [a_{ij}]$ is called skew- symmetric matrix if $a_{ij} = -a_{ji}$ for all *i, j*

or $A^T = -A$. *Example* : $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Note : ❑ All principal diagonal elements of a skew- symmetric matrix are always zero because for any diagonal element. $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

❑ Trace of a skew symmetric matrix is always 0.

Properties of symmetric and skew-symmetric matrices:

(i) If A is a square matrix, then $A + A^T, AA^T, A^TA$ are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.

(ii) If A is a symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^TAB$ are also symmetric matrices, where $n \in N$, $K \in R$ and B is a square matrix of order that of A

(iii) If A is a skew-symmetric matrix, then

(a) A^{2n} is a symmetric matrix for $n \in N$,

(b) A^{2n+1} is a skew-symmetric matrix for $n \in N$,

(c) kA is also skew-symmetric matrix, where $k \in R$,

(d) $B^T AB$ is also skew- symmetric matrix where B is a square matrix of order that of A.

(iv) If A, B are two symmetric matrices, then

(a) $A \pm B,\ AB + BA$ are also symmetric matrices,

(b) $AB - BA$ is a skew- symmetric matrix,

(c) AB is a symmetric matrix, when $AB = BA$.

(v) If A,B are two skew-symmetric matrices, then

(a) $A \pm B,\ AB - BA$ are skew-symmetric matrices,

(b) $AB + BA$ is a symmetric matrix.

(vi) If A a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquely be expressed as sum of a symmetric and skew-symmetric matrix *i.e.*

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right].$$

(2) **Singular and Non-singular matrix :** Any square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$. Here $|A|$ (or det(A) or simply det $|A|$ means corresponding determinant of square matrix A.

Example : $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \Rightarrow A$ is a non singular matrix.

(3) **Hermitian and skew-Hermitian matrix :** A square matrix $A = [a_{ij}]$ is said to be hermitian matrix if $a_{ij} = \bar{a}_{ji}\ \forall i.j$ *i.e.* $A = A^\theta$. *Example* : $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ are Hermitian matrices.

Note : ❑ If A is a Hermitian matrix then $a_{ii} = \bar{a}_{ii} \Rightarrow a_{ii}$ is real $\forall i$, thus every diagonal element of a Hermitian matrix must be real.

❑ A Hermitian matrix over the set of real numbers is actually a real symmetric matrix and a square matrix, $A = |a_{ij}|$ is said to be a skew-Hermitian if $a_{ij} = -\bar{a}_{ji}. \forall i, j$ *i.e.* $A^\theta = -A$.

Example : $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

❑ If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$ i.e. a_{ii} must be purely imaginary or zero.

❑ A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.

(4) **Orthogonal matrix :** A square matrix A is called orthogonal if $AA^T = I = A^T A$ *i.e.* if $A^{-1} = A^T$

Example : $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal because $A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$

In fact every unit matrix is orthogonal.

(5) **Idempotent matrix** : A square matrix A is called an idempotent matrix if $A^2 = A$.

Example : $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because $A^2 = \begin{bmatrix} 1/4+1/4 & 1/4+1/4 \\ 1/4+1/4 & 1/4+1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$.

Also, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2 = A$ and $B^2 = B$.

In fact every unit matrix is indempotent.

(6) **Involutory matrix** : A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Example : $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix because $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

In fact every unit matrix is involutory.

(7) **Nilpotent matrix** : A square matrix A is called a nilpotent matrix if there exists a $p \in N$ such that $A^p = 0$

Example : $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ (Here $P = 2$)

(8) **Unitary matrix** : A square matrix is said to be unitary, if $\bar{A}'A = I$ since $|\bar{A}'| = |A|$ and $|\bar{A}'A| = |\bar{A}'||A|$ therefore if $\bar{A}'A$=I, we have $|\bar{A}'||A| = 1$

Thus the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hence $\bar{A}'A = I \Rightarrow A\bar{A}' = I$

(9) **Periodic matrix** : A matrix A will be called a periodic matrix if $A^{k+1} = A$ where k is a positive integer. If, however k is the least positive integer for which $A^{k+1} = A$, then k is said to be the period of A.

(10) **Differentiation of a matrix** : If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A.

Example : If $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

(11) **Submatrix** : Let A be $m \times n$ matrix, then a matrix obtained by leaving some rows or columns or both, of A is called a sub matrix of A. *Example* : If $A' = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 2 \\ 2 & 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}$ are sub matrices of matrix $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 3 & 2 & 2 & 4 \\ 2 & 5 & 3 & 1 \end{bmatrix}$

(12) **Conjugate of a matrix** : The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex

numbers is called conjugate of A and is denoted by $\overline{A}$. *Example* : $A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$ then

$$\overline{A}=\begin{bmatrix}1-2i & 2+3i & 3-4i\\ 4+5i & 5-6i & 6+7i\\ 8 & 7-8i & 7\end{bmatrix}$$

Properties of conjugates :

(i) $\overline{(\overline{A})}=A$ (ii) $\overline{(A+B)}=\overline{A}+\overline{B}$

(iii) $\overline{(\alpha A)}=\overline{\alpha}\overline{A}$, α being any number (iv) $\overline{(AB)}=\overline{A}\,\overline{B}$, A and B being conformable for multiplication.

(13) **Transpose conjugate of a matrix :** The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ}. The conjugate of the transpose of A is the same as the transpose of the conjugate of A *i.e.* $\overline{(A')}=(\overline{A})'=A^{\theta}$.

If $A=[a_{ij}]_{m\times n}$ then $A^{\theta}=[b_{ji}]_{n\times m}$ where $b_{ji}=\overline{a}_{ij}$ *i.e.* the $(j,i)^{th}$ element of A^{θ} = the conjugate of $(i,j)^{th}$ element of A.

Example : If $A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$, then $A^{\theta}=\begin{bmatrix}1-2i & 4+5i & 8\\ 2+3i & 5-6i & 7-8i\\ 3-4i & 6+7i & 7\end{bmatrix}$

Properties of transpose conjugate

(i) $(A^{\theta})^{\theta}=A$ (ii) $(A+B)^{\theta}=A^{\theta}+B^{\theta}$ (iii) $(kA)^{\theta}=\overline{K}A^{\theta}$, K being any number (iv) $(AB)^{\theta}=B^{\theta}A^{\theta}$

Example: 16 The matrix $\begin{bmatrix}0 & 5 & -7\\ -5 & 0 & 11\\ 7 & -11 & 0\end{bmatrix}$ is known as **[Karnataka CET 2000]**

(a) Upper triangular matrix (b) Skew-symmetric matrix (c) Symmetric matrix (d)

Solution: (b) In a skew-symmetric matrix, $a_{ij}=-a_{ji}\ \forall\ i,j=1,2,3$ and $j=i$, $a_{ii}=-a_{ii}$ $\Rightarrow$ each $a_{ii}=0$

Hence the given matrix is skew-symmetric matrix $[\because A^T=-A]$.

Example: 17 The matrix $\begin{bmatrix}2 & \lambda & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ is non singular if **[Kurukshetra CEE 2002]**

(a) $\lambda\neq -2$ (b) $\lambda\neq 2$ (c) $\lambda\neq 3$ (d) $\lambda\neq -3$

Solution: (a) The given matrix $A=\begin{bmatrix}2 & \lambda & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ is non singular If $|A|\neq 0$

$$\Rightarrow |A|=\begin{vmatrix}2 & \lambda & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{vmatrix}\neq 0 \Rightarrow \begin{vmatrix}1 & \lambda+3 & 0\\ -1 & 3 & 4\\ 1 & -2 & -3\end{vmatrix}\neq 0\ [R_1\rightarrow R_1+R_2]$$

$$\Rightarrow |A|=\begin{bmatrix}1 & \lambda+3 & 0\\ 0 & 1 & 1\\ 0 & -\lambda-5 & -3\end{bmatrix}\neq 0 \quad , \begin{bmatrix}R_2\rightarrow & R_2+R_3\\ R_3\rightarrow & R_3-R_1\end{bmatrix}$$

$$\Rightarrow 1(-3+\lambda+5)\neq 0 \Rightarrow \lambda+2\neq 0 \Rightarrow \lambda\neq -2$$

Example: 18 The matrix $A=\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&-2\\-2&2&-1\end{bmatrix}$ is **[Kurukshetra CEE 2002]**

(a) Orthogonal (b) Involutary (c) Idempotent (d) Nilpotent

Solution: (a) Since for given $A=\frac{1}{3}\begin{vmatrix}1&2&2\\2&1&-2\\-2&2&-1\end{vmatrix}$. For orthogonal matrix $AA^T=A^TA=I_{(3\times3)}$

$\Rightarrow AA^T=\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&-2\\-2&2&-1\end{bmatrix}\begin{bmatrix}1&2&-2\\2&1&2\\2&-2&-1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}=3I$. Similarly $A^TA=3I$. Hence A is orthogonal

Example: 19 If $A=\begin{bmatrix}4&x+2\\2x-3&x+1\end{bmatrix}$ is symmetric, then x = **[Karnataka CET 1994]**

(a) 3 (b) 5 (c) 2 (d) 4

Solution: (b) For symmetric matrix, $A=A^T \Rightarrow \begin{bmatrix}4&2x-3\\x+2&x+1\end{bmatrix}=\begin{bmatrix}4&x+2\\2x-3&x+1\end{bmatrix} \Rightarrow 2x-3=x+2 \Rightarrow x=5$

Example: 20 If A and B are square matrices of order $n\times n$, then $(A-B)^2$ is equal to **[Karnataka CET 1999; Kerala (Engg.) 2002]**

(a) A^2-B^2 (b) $A^2-2AB+B^2$ (c) $A^2+2AB+B^2$ (d) $A^2-AB-BA+B^2$

Solution: (d) Given A and B are square matrices of order $n\times n$ we know that $(A-B)^2=(A-B)(A-B)=A^2-AB-BA+B^2$

Example: 21 If $A=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$, then which of the following statement is not correct **[DCE 2001]**

(a) A is orthogonal matrix (b) A^T is orthogonal matrix (c) Determinant $A=1$ (d)

Solution: (d) $|A|=1\neq0$, therefore A is invertible. Thus (d) is not correct

Example: 22 Matrix A is such that $A^2=2A-I$ where I is the identity matrix. Then for $n\geq2$, A^n =

(a) $nA-(n-1)I$ (b) $nA-I$ (c) $2^{n-1}A-(n-1)I$ (d) $2^{n-1}A-I$

Solution: (a) We have, $A^2=2A-I \Rightarrow A^2.A=(2A-I)A$; $A^3=2A^2-IA=2[2A-I]-IA \Rightarrow A^3=3A-2I$

Similarly $A^4=4A-3I$ and hence $A^n=nA-(n-1)I$

Example: 23 Let $A=\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}$, the only correct statement about the matrix A is **[AIEEE 2004]**

(a) $A^2=I$ (b) $A=(-1)I$, where I is unit matrix

(c) A^{-1} does not exist (d) A is zero matrix

Solution: (a) $A^2=A.A=\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$. Also, A^{-1} exists as $|A|=1$

8.2.13 Adjoint of a Square Matrix

Let $A=[a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by *adj* A

Thus, $adj\ A = [C_{ij}]^T \Rightarrow (adj\,A)_{ij} = C_{ji} =$ cofactor of a_{ji} in A.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\,A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$;

Where C_{ij} denotes the cofactor of a_{ij} in A.

Example : $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, C_{11} = s, C_{12} = -r, C_{21} = -q, C_{22} = p$

$\therefore adj\,A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

Note : ❑ The adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off diagonal elements.

Properties of adjoint matrix : If *A*, *B* are square matrices of order *n* and I_n is corresponding unit matrix, then (i) $A(adj\ A) = |A|\ I_n = (adj\ A)A$ (Thus *A* (*adj A*) is always a scalar matrix)

(ii) $|adj\ A| = |A|^{n-1}$

(iii) $adj(adj\ A) = |A|^{n-2}\ A$

(iv) $|adj\ (adj\ A)| = |A|^{(n-1)^2}$

(v) $adj(A^T) = (adj\ A)^T$

(vi) $adj\ (AB) = (adj\ B)(adj\ A)$

(vii) $adj(A^m) = (adj\ A)^m, m \in N$

(viii) $adj(kA) = k^{n-1}(adj\ A), k \in R$

(ix) $adj\ (I_n) = I_n$

(x) $adj\ (O) = O$

(xi) *A* is symmetric $\Rightarrow$ *adj A* is also symmetric.

(xii) *A* is diagonal $\Rightarrow$ *adj A* is also diagonal.

(xiii) *A* is triangular $\Rightarrow$ *adj A* is also triangular.

(xiv) *A* is singular $\Rightarrow$ |*adj A*|= 0

8.2.14 Inverse of a Matrix

A non-singular square matrix of order *n* is invertible if there exists a square matrix *B* of the same order such that $AB = I_n = BA$.

In such a case, we say that the inverse of *A* is *B* and we write $A^{-1} = B$

The inverse of A is given by $A^{-1} = \frac{1}{|A|}.adj\ A$

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$

Properties of inverse matrix:

If *A* and *B* are invertible matrices of the same order, then

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^k)^{-1} = (A^{-1})^k, k \in N$ [In particular $(A^2)^{-1} = (A^{-1})^2$]

(v) $adj(A^{-1}) = (adj\ A)^{-1}$

(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(vii) *A* = diag $(a_1 a_2 ... a_n) \Rightarrow A^{-1} = diag(a_1^{-1} a_2^{-1} ... a_n^{-1})$

(viii) *A* is symmetric $\Rightarrow A^{-1}$ is also symmetric.

(ix) *A* is diagonal, $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal.

(x) *A* is scalar matrix $\Rightarrow A^{-1}$ is also scalar matrix.

(xi) *A* is triangular, $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular.

(xii) Every invertible matrix possesses a unique inverse.

Note : ❑ (Cancellation law with respect to multiplication)

If *A* is a non singular matrix i.e., if $|A| \neq 0$, then A^{-1} exists and $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow IB = IC \Rightarrow B = C \therefore AB = AC \Rightarrow B = C \Leftrightarrow |A| \neq 0$

Example: 24 If $A = \begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}$, then $|adj\ A|$ is equal to **[UPSEAT 2003]**

(a) 16 (b) 10 (c) 6 (d) None of these

Solution: (b) $adj\ A = \begin{bmatrix} 4 & -2 \\ -3 & 4 \end{bmatrix}$

$|adj\ A| = \begin{vmatrix} 4 & -2 \\ -3 & 4 \end{vmatrix} = 16 - 6 = 10$

Example: 25 If 3, – 2 are the Eigen values of non-singular matrix *A* and |*A*| = 4 . Then Eigen values of *adj* (*A*) are **[Kurukshetra CEE 2002]**

(a) 3/4, –1/2 (b) 4/3, –2 (c) 12, –8 (d) –12, 8

Solution: (b) Since $A^{-1} = \frac{adj\ A}{|A|}$ and if λ is Eigen value of *A* then λ^{-1} is Eigen value of A^{-1}, thus for $adj\,(A)\,x = (A^{-1}x)|A|$

$= |A| .\lambda^{-1} I$

$adj(A)$ corresponding to Eigen value

$\lambda = 3$ is = 4/3 and for $\lambda = -2$ is = $4/-2$ = –2

Example: 26 If matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{K} adj\,(A)$, then *K* is **[UPSEAT 2002]**

(a) 7 (b) –7 (c) 1/7 (d) 11

Solution: (d) We know that $A^{-1} = \frac{adj(A)}{|A|}$. We have $A^{-1} = \frac{1}{K} adj(A)$ *i.e.* $K = |A|$

and $K = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3(3) - 2(1) + 4(1) = 9 - 2 + 4 = 11$

Example: 27 The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is **[AMU 2001]**

(a) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (b) $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (c) $\frac{1}{|A|}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} b & -a \\ d & -c \end{bmatrix}$

Solution: (b) Here $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, $adj\,(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Hence $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example: 28 Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10.B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If *B* is the inverse of matrix *A*, then α is **[AIEEE 2004]**

(a) 5 (b) –1 (c) 2 (d) –2

Solution: (a) We have, $A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $\therefore |A| = 1(4) + 1(5) + 1(1) = 10$ and $adj\,(A) = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

Then $A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

According to question, *B* is the inverse of matrix *A*. Hence $\alpha = 5$

Example: 29 Matrix $A = \begin{bmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{bmatrix}$ is invertible for **[UPSEAT 2002]**

(a) $K = 1$ (b) $K = -1$ (c) $K = 0$ (d) All real *K*

Solution: (d) For invertible, $|A| \neq 0$ *i.e.*, $\begin{vmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{vmatrix} \neq 0$

$\Rightarrow 1(1)-K(-K)\neq 0 \Rightarrow |A| = K^2+1 \neq 0$, which is true for all real K.

Example: 30 Let p be a non-singular matrix, **[Orissa JEE 2002]**

$1+p+p^2+.......+p^n=0$ (0 denotes the null matrix), then $p^{-1}=$

(a) p^n (b) $-p^n$ (c) $-(1+p+.....+p^n)$ (d) None of these

Solution: (a) We have, $1+p+p^2+----+p^n=0$

Multiplying both sides by p^{-1}, $p^{-1}+I+Ip+.....+p^{n-1}I=0.p^{-1}$

$p^{-1}+I(1+p+....+p^{n-1})=0 \Rightarrow p^{-1}=-I(1+p+p^2+....+p^{n-1})=-(-p^n)=p^n$.

Example: 31 Let $f(\alpha)=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$, where $\alpha\in R$,then $[f(\alpha)]^{-1}$ is equal to **[AMU 2000]**

(a) $f(-\alpha)$ (b) $f(\alpha^{-1})$ (c) $f(2\alpha)$ (d) None

Solution: (a) $|f(\alpha)|=\begin{vmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{vmatrix}=1$, *adj* of $f(\alpha)=\begin{vmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{vmatrix}$

$[f(\alpha)]^{-1}=\begin{vmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{vmatrix}$ ......(i) and $f(-\alpha)=\begin{vmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{vmatrix}$ ......(ii)

From (i) and (ii), $[f(\alpha)]^{-1}=f[-\alpha]$

Example: 32 If *I* is a unit matrix of order 10, then the determinant of *I* is equal to **[Kerala (Engg.) 2002]**

(a) 10 (b) 1 (c) 1/10 (d) 9

Solution: (b) Determinant of unit matrix of any order =1.

Example: 33 If $A=\begin{bmatrix}\alpha & 2\\ 2 & \alpha\end{bmatrix}$ and $|A^3|=125$ then $\alpha=$ **[IIT Screening 2004]**

(a) ± 3 (b) ± 2 (c) ± 5 (d) 0

Solution: (a) $125=|A^3|=|A|^3 \Rightarrow |A|=5$ and $|A|=\alpha^2-4=5 \Rightarrow \alpha^2=9 \Rightarrow \alpha=\pm 3$

Example: 34 If |*A*| denotes the value of the determinant of the square matrix *A* of order 3, then $|-2A|=$ **[MP PET 1987, 89, 92, 2000]**

(a) $-8|A|$ (b) $8|A|$ (c) $-2|A|$ (d) None of these

Solution: (a) We know that, det. $(-A)=(-1)^n \det A$, where *n* is order of square matrix

If *A* is square matrix of order 3, Then $n=3$. Hence $|-2A|=(-2)^3|A|=-8|A|$.

Example: 35 For how many value (*s*) of *x* in the closed interval [–4, –1] is the matrix $\begin{bmatrix}3 & -1+x & 2\\ 3 & -1 & x+2\\ x+3 & -1 & 2\end{bmatrix}$ singular

[Karnataka CET 2002]

(a) 2 (b) 0 (c) 3 (d) 1

Solution: (d) $\begin{vmatrix}3 & x-1 & 2\\ 3 & -1 & x+2\\ x+3 & -1 & 2\end{vmatrix}=0$

$\begin{vmatrix}0 & x & -x\\ 3 & -1 & x+2\\ x+3 & -1 & 2\end{vmatrix}=0$ $[R_1\to R_1-R_2]$, $\begin{vmatrix}0 & x & -x\\ -x & 0 & x\\ x+3 & -1 & 2\end{vmatrix}=0$ $[R_2\to R_2-R_3]$

$\begin{vmatrix}0 & 0 & -x\\ -x & x & x\\ x+3 & 1 & 2\end{vmatrix}=0$ $[C_2\to C_2+C_3]$

$-x[(-x)-x(x+3)]=0 \Rightarrow x(x^2+4x)=0 \Rightarrow x=0,-4$

Hence only one value of x in closed interval [–4,–1] i.e. $x = -4$

Example: 36 Inverse of diagonal matrix (if it exists) is a

(a) Skew-symmetric matrix (b) Diagonal matrix (c) Non invertible matrix (d) None of these

Solution: (b) Let $A = \text{diag}(d_1, d_2, d_3......, d_n)$

As A is invertible, therefore $\det(A) \neq 0 \Rightarrow d_1, d_2, d_3......., d_n \neq 0 \Rightarrow d_i \neq 0$ for i = 1, 2, 3…..n

Here, cofactor of each non diagonal entry is 0 and cofactor of a_{ii}

$$= (-1)^{i+1} \det [diag(d_1, d_2, d_3....., d_{i-1}, d_{i+1}....., d_n)] = d_1, d_2, d_3.....d_{i-1}.d_{i+1}......., d_n = \frac{1}{d_i}[d_1, d_2, d_3....., d_{i-1}, d_i, d_{i+1}....., d_n] = \frac{|A|}{d_i}$$

$$A^{-1} = \frac{1}{|A|}(adj\ A) = diag\left(\frac{1}{d_1}, \frac{1}{d_2}......., \frac{1}{d_n}\right),$$ which is a diagonal matrix

8.2.15 Elementary Transformations or Elementary Operations of a Matrix

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations

(1) Interchange of any two rows (columns)

If i^{th} row (column) of a matrix is interchanged with the j^{th} row (column), it will be denoted by $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$

Example : $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying $R_2 \leftrightarrow R_3$, we get $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

(2) Multiplying all elements of a row (column) of a matrix by a non-zero scalar

If the elements of i^{th} row (column) are multiplied by a non-zero scalar k, it will be denoted by $R_i \to R_i(k), [C_i \to C_i(k)]$ or $R_i \to kR_i, [C_i \to kC_i]$

If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying $R_2 \to 3R_2$ we obtain $B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 6 \\ -1 & 2 & -3 \end{bmatrix}$

(3) Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar k. If k times the elements of j^{th} row (column) are added to the corresponding elements of the i^{th} row (column), it will be denoted by $R_i \to R_i + kR_j (C_i \to C_i + kC_j)$

If $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then the application of elementary operation $R_3 \to R_3 + 2R_1$ gives the matrix

$B = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 4 & 3 & 9 & 3 \end{bmatrix}$, If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim B$

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}$, then $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$, applying $R_2 \to R_2 + (-1)R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & 1 & -2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$, applying $C_4 \to C_4 + (-1)C_3$

An elementary transformation is called a row transformation or a column transformation according as it is applied to rows or columns.

8.2.16 Elementary Martix

A matrix obtained from an identity matrix by a single elementary operation (transformation) is called an elementary matrix. *Example :* $\begin{bmatrix}1 & 3 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ are elementary matrices obtained from I_3 by subjecting it to the elementary transformations $R_1 \to R_1 + 3R_2,\ C_1 \leftrightarrow C_3$ and $R_2 \leftrightarrow R_3$ respectively.

Theorem 1 : Every elementary row (column) transformation of an *m*×*n* matrix (not identity matrix) can be obtained by pre-multiplication (post- multiplication) with the corresponding elementary matrix obtained from the identity matrix $I_m\ (I_n)$ by subjecting it to the same elementary row (column) transformation.

Theorem 2 : Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of *AB* can be obtained by subjecting the pre-factor *A* (post factor B) to the same elementary row (column) transformation.

Method of finding the inverse of a matrix by elementary transformations : Let A be a non singular matrix of order *n*. Then *A* can be reduced to the identity matrix I_n by a finite sequence of elementary transformation only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore there exist elementary matrices $E_1, E_2.....E_k$ such that

$$(E_k E_{k-1...} E_2 E_1)A = I_n$$

$$\Rightarrow (E_k E_{k-1}...E_2 E_1)AA^{-1} = I_n A^{-1} \quad \text{(post multiplying by } A^{-1})$$

$$\Rightarrow (E_k E_{k-1}...E_2 E_1)I_n = A^{-1} \qquad (\because I_n A^{-1} = A^{-1} \text{and } AA^{-1} = I_n) \Rightarrow A^{-1} = (E_k E_{k-1}...E_2 E_1)I_n$$

Algorithm for finding the inverse of a non singular matrix by elementary row transformations

Let *A* be non- singular matrix of order *n*

Step I : Write $A = I_n A$

Step II : Perform a sequence of elementary row operations successively on A on the LHS and the pre factor I_n on the RHS till we obtain the result $I_n = BA$

Step III : Write $A^{-1} = B$

Note : ❑ The following steps will be helpful to find the inverse of a square matrix of order 3 by using elementary row transformations.

Step I : Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to first row.

Step II : After introducing unity at (1,1) place introduce zeros at all other places in first column.

Step III Introduce unity at the intersection of 2nd row and 2nd column with the help of 2nd and 3rd row.

Step IV : Introduce zeros at all other places in the second column except at the intersection of 2nd row and 2nd column.

Step V : Introduce unity at the intersection of 3rd row and third column.

Step VI : Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.

Example: 37 Using elementary row transformation find the inverse of the matrix $A = \begin{bmatrix}3 & -1 & -2\\2 & 0 & -1\\3 & -5 & 0\end{bmatrix}$

(a) $\begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$ (b) $\frac{1}{8}\begin{bmatrix} 5 & -5 & 1 \\ 3 & -3 & 1 \\ 0 & 3 & 1 \end{bmatrix}$ (c) $\frac{1}{8}\begin{bmatrix} 5 & 5 & 1 \\ 3 & 6 & -1 \\ 10 & -12 & 2 \end{bmatrix}$ (d) None of these

Solution: (a) We have *A*=*IA* ⇒ $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $(R_1 \to R_1 - R_2)$ $\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$

Applying $R_2 \to R_2 / 2$, $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 2R_2$, $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5 & 6 & 1 \end{bmatrix} A$

Applying $R_3 \to R_3 / 4$, $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ 5/4 & 6/4 & 1/4 \end{bmatrix} A$

Applying $R_1 \to R_1 + \frac{1}{2}R_3$ and $R_2 \to R_2 - \frac{1}{2}R_3$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$

$$A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$$

8.2.17 Rank of Matrix

Definition : Let *A* be a *m*×*n* matrix. If we retain any *r* rows and *r* columns of *A* we shall have a square sub-matrix of order *r*. The determinant of the square sub-matrix of order *r* is called a minor of *A* order *r*.

Consider any matrix *A* which is of the order of 3×4 say, $A = \begin{vmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{vmatrix}$. It is 3×4 matrix so we can have minors of order 3, 2 or 1. Taking any three rows and three columns minor of order three. Hence minor of order

$$3 = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix} = 0$$

Making two zeros and expanding above minor is zero. Similarly we can consider any other minor of order 3 and it can be shown to be zero. Minor of order 2 is obtained by taking any two rows and any two columns.

Minor of order $2 = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1 \neq 0$. Minor of order 1 is every element of the matrix.

Rank of a matrix: The rank of a given matrix *A* is said to be *r* if

(1) Every minor of *A* of order *r*+1 is zero

(2) There is at least one minor of *A* of order *r* which does not vanish

Note : ❑ If a minor of A is zero the corresponding submatrix is singular and if a minor of *A* is not zero then corresponding submatrix is non-singular.

Here we can also say that the rank of a matrix *A* is said to be *r* if

(i) Every square submatrix of order *r*+1 is singular.

(ii) There is at least one square submatrix of order *r* which is non-singular.

The rank *r* of matrix *A* is written as $\rho(A)=r$

Working rule : Calculate the minors of highest possible order of a given matrix A. If it is not zero, then the order of the minor is the rank. If it is zero and all other minors of the same order be also zero, then calculate minor of next lower order and if at least one of them is not zero then this next lower order will be the rank. If, however, all the minors of next lower orders are zero, then calculate minors of still next lower order and so on.

Note : ❑ The rank of the null matrix is not defined and the rank of every non-null matrix is greater than or equal to 1.

❑ The rank of a singular square matrix of order *n* cannot be *n*.

8.2.18 Echelon form of a Matrix

A matrix *A* is said to be in Echelon form if either *A* is the null matrix or *A* satisfies the following conditions:

(1) Every non- zero row in *A* precedes every zero row.

(2) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

It can be easily proved that the rank of a matrix in Echelon form is equal to the number of non-zero row of the matrix. *Example* : The rank of the matrix $A=\begin{bmatrix}0&3&2&1\\0&0&2&5\\0&0&0&0\end{bmatrix}$ is 2 because it is in Echelon form and it has two non-zero rows. The matrix $A=\begin{bmatrix}0&2&5\\0&0&1\\0&0&-4\end{bmatrix}$ is not in Echelon form, because the number of zeros in second row is not less than the number of zeros in the third row. To reduce A in the echelon form, we apply some elementary row transformations on it. Applying $R_3\to R_3+4R_2$, we obtain $A\sim\begin{bmatrix}0&2&5\\0&0&1\\0&0&0\end{bmatrix}$, which is in Echelon form and contains 2 non zero rows. Hence, $r(A)=2$

Rank of a matrix in Echelon form : The rank of a matrix in Echelon form is equal to. the number of non-zero rows in that matrix.

Algorithm for finding the rank of a matrix : Let $A=[a_{ij}]$ be an *m*×*n* matrix.

Step I : Using elementary row transformations make $a_{11}=1$

Step II : Make $a_{21},a_{31},....,a_{m1}$ all zeros by using elementary transformations,

$R_2\to R_2-a_{21}R_1, R_3\to R_3-a_{31}R_1,.....R_m\to R_m-a_{m1}R_1$

Step III : Make $a_{22}=1$ by using elementary row transformations.

Step IV : Make $a_{32}, a_{42}, ..., a_{m2}$ all zeros by using $R_3 \to R_3 - a_{32}R_2, R_4 \to R_4 - a_{42}R_2, ...R_m \to R_m - a_{m2}R_2$

The process used in steps III and IV is repeated upto $(m-1)th$ row. Finally we obtain a matrix in Echelon form, which is equivalent to the matrix *A*. The rank of *A* will be equal to the number of non-zeros rows in it.

Example: 38 The rank of the matrix $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}$ is **[Kurukshetra CEE 2002]**

(a) 2 (b) 3 (c) 1 (d) Indeterminate

Solution: (a) We have $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix}_{3\times4}$, Considering 3×3 minor $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}_{3\times3}$ its determinant is 0.

Similarly considering , $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & -4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix}$, their determinant is 0 each rank can not be 3

Then again considering a 2×2 minor, $\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}$, which is non zero. Thus, rank =2

Example: 39 The rank of the matrix $\begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix}$ is **[Roorkee 1988]**

(a) 1 if *a* = 6 (b) 2 if *a* =1 (c) 3 if *a* = 2 (d) 1 if *a* = – 6

Solution: (b,d) Let $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix} = \begin{vmatrix} 0 & 0 & a+6 \\ 0 & 0 & -a-6 \\ 1 & -2 & a+1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -a-6 \\ 1 & -2 & a+1 \end{vmatrix}$

When $a = -6$, $A = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & -5 \end{vmatrix}$, $\therefore r(A) = 1$

When $a = 1$, $A = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -7 \\ 1 & -2 & 2 \end{vmatrix}$ $\therefore r(A) = 2$, When $a = 6$, $A = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -12 \\ 1 & -2 & 7 \end{vmatrix}$, $\therefore r(A) = 2$

When $a = 2$, $A = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -8 \\ 1 & -2 & 3 \end{vmatrix}$, $\therefore r(A) = 2$

Example: 40 The value of x so that the matrix $\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$ has rank 3 is

(a) $x \neq 0$ (b) $x = a+b+c$
(c) $x \neq 0$ and $x \neq -(a+b+c)$ (d) $x = 0,\ x = a+b+c$

Solution: (c) Since rank is 3, $|A|_{3\times3} \neq 0$, $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}_{3\times3} \neq 0$

$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix} \neq 0$, Applying $(C_1 \to C_1 + C_2 + C_3)$

$$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix} \neq 0 \Rightarrow x+a+b+c \neq 0, \begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix} \neq 0$$

$$x \neq -(a+b+c), \begin{vmatrix} 0 & -x & c \\ 0 & x & -x \\ 1 & b & x+c \end{vmatrix} \neq 0 \Rightarrow x \neq 0$$

8.2.19 System of Simultaneous Linear Equations

Consider the following system of *m* linear equations in *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

The system of equations can be written in matrix form as $\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ or $AX = B$,

Where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

The *m*×*n* matrix A is called the coefficient matrix of the system of linear equations.

(1) **Solution :** A set of values of the variables $x_1, x_2 \ldots\ldots x_n$ which simultaneously satisfy all the equations is called a solution of the system of equations. *Example* : $x = 2, y = -3$ is a solution of the system of linear equations $3x + y = 3$, $2x + y = 1$, because 3(2) +(–3)=3 and 2(2)+(–3)=1

(2) **Consistent system :** If the system of equations has one or more solutions, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations. *Example* : the system of linear equation $2x + 3y = 5$, $4x + 6y = 10$ is consistent, because *x*=1, *y* =1 and $x = 2, y = 1/3$ are solutions of it.

However, the system of linear equations $2x + 3y = 5$, $4x + 6y = 10$ is inconsistent, because there is no set of values of *x, y* which satisfy the two equations simultaneously.

(3) **Homogeneous and non-homogeneous system of linear equations:** A system of equations *AX=B* is called a homogeneous system if $B = 0$. Otherwise, it is called a non-homogeneous system of equations.

Example : The system of equations, $2x + 3y = 0$, $3x - y = 5$ is a homogeneous system of linear equations whereas the system of equations given by $2x + 3y = 1$, $3x - y = 5$ is a non homogeneous system of linear equations.

8.2.20 Solution of a Non Homogeneous System of Linear Equations

There are three methods of solving a non homogeneous system of simultaneous linear equations.

(1) Determinant Method (Cramer's Rule) (2) Matrix method (3) Rank method

We have already discussed the determinant method (Cramer's rule) in chapter determinants.

(1) **Matrix method :** Let $AX = B$ be a system of *n* linear equations with *n* unknowns. If *A* is non-singular, then A^{-1} exists. $\therefore AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$, [pre-multiplying by A^{-1}]

$\Rightarrow (A^{-1}A)X = A^{-1}B$, [by associativity]

$\Rightarrow I_n X = A^{-1}B \Rightarrow X = A^{-1}B$. Thus, the system of equations $AX = B$ has a solution given by $X = A^{-1}B$.

Now, let X_1 and X_2 be two solutions of $AX = B$. then, $AX_1 = B$ and $AX_2 = B$

$\Rightarrow AX_1 = AX_2 \Rightarrow A^{-1}(AX_1) = A^{-1}(AX_2) \Rightarrow (A^{-1}A)X_1 = (A^{-1}A)X_2 \Rightarrow I_n X_1 = I_n X_2 \Rightarrow X_1 = X_2$.

Hence, the given system has a unique solution.

Thus, if *A* is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.

If *A* is a singular matrix, then the system of equations given by *AX=B* may be consistent with infinitely many solutions or it may be inconsistent also.

Criterion of consistency : Let $AX = B$ be a system of *n*-linear equations in *n* unknowns.

(i) If $|A| \neq 0$, then the system is consistent and has a unique solution given by $X = A^{-1}B$

(ii) If $|A| = 0$ and (*adj A*) $B = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and $(adj\ A)\ B \neq 0$, then the system is inconsistent

Algorithm for solving a non-homogeneous system of linear equations : We shall give the algorithm for three equations in three unknowns. But it can be generalized to any number of equations.

Let $AX = B$ be a non-homogenous system of 3 linear equations in 3 unknowns. To solve this system of equations we proceed as follows

Step I : Write the given system of equations in matrix form, $AX = B$ and obtain A, B.

Step II : Find $|A|$

Step III : If $|A| \neq 0$, then write "the system is consistent with unique solution". obtain the unique solution by the following procedure. Find A^{-1} by using $A^{-1} = \frac{1}{|A|} adj A$ obtain the unique solution given by $X = A^{-1}B$

Step IV : If $|A| = 0$, then write "the system is either consistent with infinitely many solutions or it is inconsistent. To distinguish these two, proceed as follows: Find (*adj A*) *B.*

If $(adj\ A)B \neq 0$, then write "the system is inconsistent".

If $(adj\ A)B = 0$, then the system is consistent with infinitely many solution. To find these solutions proceed as follows. Put $z = k$ (any real number) and take any two equations out of three equations. Solve these equations for *x* and *y.* Let the values of *x* and *y* be λ and μ respectively. Then $x = \lambda$, $y = \mu$ $z = k$ is the required solution, where any two of λ, μ, k are functions of the third.

(2) **Rank method :** Consider a system of *m* simultaneous linear equations in *n* unknowns $x_1, x_2 ...., x_n$, given by $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x = b_1$

$$\begin{matrix} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m \end{matrix}$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $AX = B$,where $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

The matrix *A* is called the coefficient matrix and the matrix

$[A:B] = \left[\begin{array}{ccccc:c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$ is called the augmented matrix of the given system of equations. This matrix is obtained by adding $(n+1)$ column to *A.* The elements of this column are $b_1, b_2 ...., b_m$

For example, the augmented matrix of the system of equation

$$2x - y + 3z = 1$$
$$x + y - 2z = 5$$
$$x + y + z = -1$$ is

$$\left[\begin{array}{ccc:c} 2 & -1 & 3 & 1 \\ 1 & 1 & -2 & 5 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

A non-homogeneous system of linear equations may have a unique solution, or many solutions or no solution at all. If it has a solution (whether unique or not) the system is said to be consistent. Otherwise it is called an inconsistent system. The following theorems tell us about the condition for consistency of a system of linear equations

Theorem 1 : The system of linear equations $AX = B$ is consistent iff the rank of the augmented matrix $[A:B]$ is equal to the rank of the coefficient matrix *A.*

Theorem 2 : Let $AX = B$ be a system of *m* simultaneous linear equations in *n* unknowns.

Case I : If $m > n$, then

(i) if $r(A) = r(A:B) = n,$ then system of linear equations has a unique solution.

(ii) if $r(A) = r(A:B) = r < n,$ then system of linear equations is consistent and has infinite number of solutions. In fact, in this case $(n-r)$variables can be assigned arbitrary values.

(iii) if $r(A) \neq r(A:B),$ then the system of linear equations is inconsistent i.e. it has no solution.

Case II : If $m < n$ and $r(A) = r(A:B) = r,$ then $r \le m < n$ and so from (ii) in case I, there are infinite number of solutions.

Thus, when the number of equations is less than the number of unknowns and the system is consistent, then the system of equations will always have an infinite number of solutions.

Algorithm for solving a non-homogeneous system *AX=B* of linear equations by rank method

Step I: Obtain *A*, B.

Step II : Write the Augmented matrix [A : B].

Step III : Reduce the augmented matrix to Echelon form by applying a sequence of elementary row-operations.

Step IV : Determine the number of non-zero rows in *A* and [*A* : *B*] to determine the ranks of A and [*A* :*B*] respectively.

Step V: If $r(A) \neq r(A:B)$ then write "the system is inconsistent" STOP else write "the system is consistent", go to Step VI

Step VI : If $r(A) = r(A:B) =$ number of unknowns, then the system has a unique solution which can be obtained by back substitution.

If $r(A) = r(A:B) <$ number of unknowns, then the system has an infinite number of solutions which can also be obtained by back substitution.

Example: 41 If $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$ then values of x, y, z, w are

(a) 2, 2, 3, 4 (b) 2, 3, 1,2 (c) 3, 3, 0, 1 (d) None of these

Solution: (a) We have $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$

$x+y=4,\ \ 2x+z=7,\ x-y=0$ and $2z+\omega=10 \Rightarrow x=2$ and $y=2,\ \ z=3,\ \ w=4$

Example: 42 The system of linear equation $x+y+z=2$, $2x+y-z=3$, $3x+2y+kz=4$ has unique solution if **[EAMCET 1994]**

(a) $K \neq 0$ (b) $-1 < K < 1$ (c) $-2 < K < 2$ (d) $K = 0$

Solution: (a) The given system of equation has a unique solution if $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow K \neq 0$

8.2.21 Cayley-Hamilton Theorem

Every matrix satisfies its characteristic equation e.g. let *A* be a square matrix then $|A - xI| = 0$ is the characteristics equation of A. If $x^3 - 4x^2 - 5x - 7 = 0$ is the characteristic equation for *A*, then

$A^3 - 4A^2 + 5A - 7I = 0$

Roots of characteristic equation for *A* are called Eigen values of A or characteristic roots of *A* or latent roots of *A.*

If λ is characteristic root of *A*, then λ^{-1} is characteristic root of A^{-1}.

8.2.22 Geometrical Transformations

(1) **Reflexion in the *x*-axis:** If $P'(x', y')$ is the reflexion of the point $P(x, y)$ on the x-axis, then the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflexion of a point $P(x, y)$ in the *x*-axis.

(2) **Reflexion in the *y*-axis :** Here the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(3) **Reflexion through the origin :** Here the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(4) **Reflexion in the line $y = x$:** Here the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(5) **Reflexion in the line *y* = – *x* :** Here the matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(6) **Reflexion in y = *x* tan θ :** Here matrix is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

(7) **Rotation through an angle** θ **:** Here matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

8.2.23 Matrices of Rotation of Axes

We know that if x and y axis are rotated through an angle θ about the origin the new coordinates are given by $x = X\cos\theta - Y\sin\theta$ and $y = X\sin\theta + Y\cos\theta$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the matrix of rotation through an angle θ .

Example: 43 Characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

(a) $A^3 - 20A + 8I$ (b) $A^3 + 20A + 8I$ (c) $A^3 - 80A + 20I$ (d) None of these

Solution: (a) The characteristic equation is $|A - \lambda I| = 0$.

So, $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$ i.e. $\lambda^3 - 20\lambda + 8 = 0$

By cayley-Hamilton theorem , $A^3 - 20A + 8I = 0$

Example: 44 The transformation due to the reflection of (x, y) through the origin is described by the matrix

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: (b) If (x', y') is the new position

$x' = (-1)x + 0.y,\ y' = 0.x + (-1)yd$

$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore$ Transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Example: 45 The rotation through 180^0 is identical to

(a) The reflection in x -axis (b) The reflection in y-axis (c) A point reflection (d) Identity transformation

Solution: (c) Rotation through 180^0 gives $x' = -x$

$y' = -y$. Hence this a point reflection.

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